Quantum Morphisms

Lecture 6

Last Week

- If $G \xrightarrow{q} H$ but $G \not\to H$, then $\alpha(G \ltimes H) < \alpha_q(G \ltimes H) = |V(G)|$.
- $\exists\, \gamma < 2$ s.t. $\alpha(\Omega_{4n} \square K_{4n}) < 4n\gamma^{4n}$, but $\alpha(\Omega_{4n} \square K_{4n}) = 2^{4n}$.
- $\chi_q(G) < \frac{|V(G)|}{\alpha_q(G)}$ is possible, e.g. $L(\text{Paley}(9)) \cong L(L(K_{3,3}))$.
- $\alpha_q(2G) > 2\alpha_q(G)$ is possible.
- $G \xrightarrow{q} H \Rightarrow \bar{\vartheta}(G) \leq \bar{\vartheta}(H)$ i.e. $\bar{\vartheta}$ is quantum homomorphism monotone.
- Thus $\omega_q(G) \leq \bar{\vartheta}(G) \leq \chi_q(G)$.

Remarks

- Variants ϑ^- & ϑ^+ due to Schrijver & Szegedy resp.

$$\omega(G) \leq \omega_q(G) \leq \omega_p(G) \leq \bar{\vartheta}^-(G) \leq \bar{\vartheta}(G) \leq \bar{\vartheta}^+(G) \leq \xi_f(G) \leq \chi_q(G) \leq \chi(G)$$

$$\xi(G)$$

$$\chi_f(G)$$

- For a few years the thetas were the only known efficiently computable upper/lower bounds on $\alpha_q, \omega_q, \chi_q$ etc.

- Recently, Elphick & Wocjan (and sometimes others) have shown that many known spectral bounds on classical parameters are also bounds on their quantum counterparts.

The Inertia Bound

Let $M \in \mathbb{C}^{n \times n}$ be a <u>Hermitian</u> matrix.

$n^+(M)$:= number of positive eigenvalues of M
$n^0(M)$:= " " zero " " "
$n^-(M)$:= " " negative " " "

$(n^+(M), n^0(M), n^-(M))$:= inertia of M $\quad (n = n^+ + n^0 + n^-)$

<u>Theorem (Cvetković)</u>: Let M be a Hermitian weighted adjacency matrix for G, i.e. $M_{uv} = 0$ if $u \not\sim v$. Then

$$\alpha(G) \le n^0(M) + \min\{n^+(M), n^-(M)\}.$$

Proof uses interlacing.

Hermitian

$\lambda_i^{\downarrow}(M)$:= i^{th} largest eigenvalue of M (including multiplicities)
$\lambda_i^{\uparrow}(M)$:= " smallest " " " " "

$$\lambda_1^{\downarrow}(M) \ge \lambda_2^{\downarrow}(M) \ge \dots \ge \lambda_n^{\downarrow}(M)$$
$$\lambda_1^{\uparrow}(M) \le \lambda_2^{\uparrow}(M) \le \dots \le \lambda_n^{\uparrow}(M)$$

Interlacing Theorem: Let $M \in \mathbb{C}^{n \times n}$ be Hermitian and let $R \in \mathbb{C}^{n \times k}$ be s.t. $R^* R = I \in \mathbb{C}^{k \times k}$. Then

$$\lambda_i^{\downarrow}(M) \geq \lambda_i(R^* M R) \quad \forall\, i = 1, \dots, k$$

and

$$\lambda_i^{\uparrow}(M) \leq \lambda_i(R^* M R) \quad \forall\, i = 1, \dots, k.$$

Corollary: Let $M \in \mathbb{C}^{n \times n}$ be Hermitian and let M' be a $k \times k$ principal submatrix of M. Then

$$\lambda_i^{\downarrow}(M) \geq \lambda_i^{\downarrow}(M') \quad \forall\, i = 1, \dots, k$$

and

$$\lambda_i^{\uparrow}(M) \leq \lambda_i^{\uparrow}(M') \quad \forall\, i = 1, \dots, k.$$

Proof: Let $i_1, \dots, i_k$ be the row/column indices of M'. Then $R = \sum_{j=1}^{k} |i_j \rangle\langle j| \in \mathbb{C}^{n \times k}$ satisfies $R^* R = I$ & $R^* M R = M'$.

Theorem (Cvetković): Let M be a Hermitian weighted adjacency matrix for G, i.e. $M_{uv}=0$ if $u \not\sim v$. Then

$$\alpha(G) \leq n^0(M) + \min\{n^+(M), n^-(M)\}.$$

Proof: Suppose $S \subseteq V(G)$ is an independent set of size $\alpha(G)$. Let M be a Hermitian weighted adjacency matrix for G. Let M' be the principal submatrix of M consisting of the rows/columns indexed by the elements of S. Then $M'=0$

and $\lambda_i^{\downarrow}(M) \geq \lambda_i^{\downarrow}(M') = 0$ for $i=1,\ldots,\alpha(G)$.

Thus $n^0(M) + n^+(M) \geq \alpha(G)$. $M_{uv}=0$ if $u \not\sim v$

Similarly $n^0(-M) + n^+(-M) \geq \alpha(G)$

$$\underbrace{n^0(-M) + n^+(-M)}_{\textstyle n^0(M) + n^-(M)}$$

$n^0 + \min\{n^+, n^-\}$ $\square$

Elzinga & Gregory: Can we always attain equality using some real symmetric M?

John Sinkovic (2016): No, Paley(17).

What about complex Hermitian M?

Isotropic Subspaces

Let $M \in \mathbb{C}^{n\times n}$. A subspace $U \subseteq \mathbb{C}^n$ is M-isotropic if $\langle x|M|y\rangle = 0 \ \ \forall x, y \in U$.

Lemma (Elzinga + Gregory / Elphick + Wocjan)

Let $M \in \mathbb{C}^{n\times n}$ be Hermitian. Then the maximum dimension of an M-isotropic subspace is

$$n^0(M) + \min\{n^+(M), n^-(M)\}.$$

Proof: WLOG let $n^- \leq n^+$. First we show this dimension can be attained.

$$\left.\begin{array}{ll} |\psi_i^0\rangle & i=1,\dots,n^0 \\ |\psi_i^-\rangle & i=1,\dots,n^- \\ |\psi_i^+\rangle & i=1,\dots,n^+ \end{array}\right\} \text{full set of orthonormal eigenvectors for } M \text{ with} \quad \begin{array}{l} M|\psi_i^0\rangle = 0 \\ M|\psi_i^-\rangle = \lambda_i^-|\psi_i^-\rangle,\ \lambda_i^- < 0 \\ M|\psi_i^+\rangle = \lambda_i^+|\psi_i^+\rangle,\ \lambda_i^+ > 0 \end{array}$$

Define $\overset{\neq 0}{|\varphi_i\rangle} = |\psi_i^+\rangle + \left(\frac{\lambda_i^+}{-\lambda_i^-}\right)^{1/2}|\psi_i^-\rangle$ for $i=1,\dots,n^-$

Then $\langle\varphi_i|\varphi_j\rangle = \langle\varphi_i|M|\varphi_j\rangle = 0$ if $i \neq j$

and $\langle\varphi_i|M|\varphi_i\rangle = \lambda_i^+\overset{=1}{\langle\psi_i^+|\psi_i^+\rangle} + \lambda_i^-\left(\frac{\lambda_i^+}{-\lambda_i^-}\right)\overset{=1}{\langle\psi_i^-|\psi_i^-\rangle}$

$$= \lambda_i^+ - \lambda_i^+ = 0.$$

Thus $\mathrm{span}(\{|\psi_i^0\rangle : i=1,\dots,n^0\} \cup \{|\varphi_i\rangle : i=1,\dots,n^-\})$ is an M-isotropic subspace of dimension n^0+n^-. $\left(\sum \alpha_i^* \langle\psi_i^0| + \sum \beta_i^* \langle\varphi_i|\right) M \left(\sum \alpha_i |\psi_i^0\rangle + \sum \beta_i |\varphi_i\rangle\right) = 0$

Now let U be any M-isotropic subspace and let $V = \mathrm{span}(\{|\psi_i^+\rangle : i=1,\dots,n^+\})$. Then $U \cap V = \{0\}$ and thus

$$\begin{aligned} n^+ + n^0 + n^- = n &\geq \dim(U+V) \\ &= \dim(U) + \dim(V) - \dim(U \cap V) \\ &= \dim(U) + n^+. \end{aligned}$$

Therefore $\dim(U) \leq n^0 + n^-$. $\square$

Alternative proof of inertia bound (Chris):

If $S \subseteq V(G)$ is an independent set, then $\mathrm{span}(\{|v\rangle : v \in S\})$ is an M-isotropic subspace for any weighted adjacency matrix M since

$$\langle u|M|v\rangle = M_{uv} = 0 \text{ for } u,v \in S.$$

Didn't need M to be Hermitian.

Optimized inertia bound:

$\min\{\max\{\dim(U) : U \text{ is } M\text{-isotropic}\} : M \text{ is a wt'd adj mtx}\}$

take M optimal here

Elzinga + Gregory's question: is this equal to $\alpha(G)$?

What about

$\hat{\alpha}(G) := \max\{\dim(U) : U \text{ is } M\text{-isotropic } \forall \text{ wt'd adj mts } M\}$?

take $\hat{U}$ that is optimal here

Lemma (Duan, Severini, Winter): $\hat{\alpha}(G) = \alpha(G)$.

$\alpha(G) \leq \hat{\alpha}(G)$: S max indpt set, take $U = \mathrm{span}\{|v\rangle : v \in S\}$.

$\hat{\alpha}(G) \leq \alpha(G)$: Show $\{v \in V(G) : |v\rangle \notin U^{\perp}\}$ is an independent set.

$\Rightarrow \dim(U) \leq \alpha(G)$.

Elzinga + Gregory's question: is there always a wt'd adj mtx M with maximum M-isotropic subspace U s.t. U is M'-isotropic for all wt'd adj mts M'?

Elphick + Wocjan

$E_u E_v = 0$ if $u \sim v$

Recall: $\alpha_p(G) = \sup\{\frac{1}{d}\sum_{v\in V(G)} rk(E_v) : v \mapsto E_v \in \mathbb{C}^{d\times d}$ is a proj. pack.$\}$

Theorem (Elphick + Wocjan): Let M be a Hermitian weighted adjacency matrix for G.
Then $\alpha_p(G) \le n^0(M) + \min\{n^+(M), n^-(M)\}$.

Proof: Let $v \mapsto E_v \in \mathbb{C}^{d\times d}$ be a proj. pack.
For each $v \in V(G)$, spectrally decompose E_v as

$$E_v = \sum_{i=1}^{r_v} |\psi_i^v\rangle\langle\psi_i^v| \quad \text{where } r_v = rk(E_v).$$

Define

$$|\varphi_i^v\rangle = |v\rangle \otimes |\psi_i^v\rangle \in \mathbb{C}^{V(G)} \otimes \mathbb{C}^d.$$

Then

$$\langle\varphi_i^v|\varphi_j^u\rangle = \langle v|u\rangle\langle\psi_i^v|\psi_j^u\rangle = 0 \text{ unless } v=u \text{ + } i=j,$$

and

$$\langle\varphi_i^v|(M\otimes I_d)|\varphi_j^u\rangle = \underbrace{\langle v|M|u\rangle}_{\substack{=0 \\ \text{unless } v\sim u}} \underbrace{\langle\psi_i^v|\psi_j^u\rangle}_{\substack{=0 \\ \text{if } v\sim u}} = 0.$$

Therefore, $\mathrm{span}\{|\varphi_i^v\rangle : v \in V(G),\ i \in [r_v]\}$ is an $M \otimes I_d$-isotropic subspace of dimension $\sum_v rk(E_v)$.

Thus,
$$\sum_v rk(E_v) \le n^0(M \otimes I_d) + \min\{n^+(M \otimes I_d), n^-(M \otimes I_d)\}$$
$$= d\left(n^0(M) + \min\{n^+(M), n^-(M)\}\right).$$

$\Rightarrow$ $\frac{1}{d}\sum_v rk(E_v) \le n^0(M) + \min\{n^+(M), n^-(M)\}$. $\square$

Corollary: There is no Hermitian weighted adjacency matrix M for $L(Paley(9))$ with
$$\alpha(L(Paley(9))) = n^0(M) + \min\{n^+(M), n^-(M)\}.$$

Say that a matrix *fits* G if $M_{ij}=0$ whenever $i\neq j$ & $i\not\sim j$.

Proposition (Duan, Severini, Winter):

$$\alpha(G)=\max\left\{ r\in\mathbb{N} \;\middle|\; \begin{array}{l} \exists\, |\psi_1\rangle,\dots,|\psi_r\rangle\in\mathbb{C}^{V(G)} \text{ s.t. } \langle\psi_i|\psi_j\rangle=\delta_{ij} \\ \text{and } \langle\psi_i|M|\psi_j\rangle=0 \text{ if } i\neq j \text{ for all} \\ M \text{ that fit } G. \end{array} \right\}$$

Proof: Exercise.

Theorem:

$$\alpha_p(G)=\sup_d\left\{\frac{1}{d}\max\left\{ r \;\middle|\; \begin{array}{l} \exists\, |\psi_1\rangle,\dots,|\psi_r\rangle\in\mathbb{C}^{V(G)}\otimes\mathbb{C}^d \text{ s.t. } \langle\psi_i|\psi_j\rangle=\delta_{ij} \\ \text{and } \langle\psi_i|M\otimes I_d|\psi_j\rangle=0 \text{ if } i\neq j \\ \text{for all } M \text{ that fit } G. \end{array} \right\}\right\}$$

Proof: Exercise.

The max in the above expression for $\alpha_p(G)$ is equal to the one-shot zero-error classical capacity of the quantum channel consisting of a noiseless quantum channel of dimension d and a noisy classical channel with confusability graph G. See arXiv:1002.2514 for definitions.